

Change of variable and the rapidity of decrease of Fourier coefficients

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Abstract. We consider the class $C(T)$ of continuous real-valued functions on the circle. For certain classes of functions naturally characterised by the rapidity of decrease of Fourier coefficients we investigate whether it is possible to bring families of functions in $C(T)$ into these classes by a change of variable. This paper was originally published in *Matematicheskii Sbornik*, **181**:8 (1990), 1099–1113 (Russian). The English translation, published in *Mathematics of the USSR, Sbornik*, **70**:2 (1991), 541–555, is to a large extent inconsistent with the original text. Herein the author provides a corrected translation.

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Introduction

We consider the class $C(T)$ of continuous real-valued functions on the circle T , and the Fourier series of functions in $C(T)$:

$$f \sim \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{ikt} \quad (1)$$

(\mathbb{Z} is the set of integers).

It is known that certain properties of functions of class $C(T)$, related to the series (1), can be improved by a change of variable, i.e., by a homeomorphism of T onto itself. For a survey of basic results in the area and a number of open problems see the papers by A. M. Olevskii [1], [2]. Following [1] and [2] we quote some of these results.

THEOREM A (J. Pál, 1914; H. Bohr, 1935). *For every function $f \in C(T)$ there exists a homeomorphism h of T onto itself such that the superposition $f \circ h$ belongs to the class $U(T)$ of functions with uniformly convergent Fourier series.*

The method used to prove this theorem allows us to obtain the rapid decrease of the Fourier coefficients, namely

$$|\widehat{f \circ h}(k)| = d(k) + O(1/|k|), \quad |k| \rightarrow \infty,$$

where

$$\sum_{k \in \mathbb{Z}} |d(k)|^2 |k| < \infty,$$

whence, in particular, we see that

$$\sum_k |\widehat{f \circ h}(k)|^p < \infty \quad \forall p > 1. \quad (2)$$

On the other hand, we have the following theorem that solves a problem posed by N. N. Lusin.

THEOREM B (A. M. Olevskiĭ, 1981). *There exists an $f \in C(T)$ such that there is no change of variable which will bring f into the algebra $A(T)$ of absolutely convergent Fourier series, i.e., $f \circ h \notin A(T)$ for every homeomorphism h of T onto itself.*

Thus, in general, it is impossible to attain condition (2) for $p = 1$.

On the other hand, the following theorem shows that one can approach the class $A(T)$ arbitrarily close.

THEOREM C (A. A. Saakyan, 1979). *If $\alpha(n), n = 0, 1, 2, \dots$, is a positive sequence satisfying the condition $\sum_n \alpha(n) = \infty$ and a certain condition of regularity, then for every $f \in C(\mathbb{T})$ there is a homeomorphism h such that $\widehat{f \circ h}(k) = O(\alpha(|k|))$ (see [1], Theorem 4.1).*

J.-P. Kahane and Y. Katznelson investigated whether it is possible to bring families of functions into the class $U(T)$. They obtained the following result (1978).

THEOREM D. *For every compact set K in $C(T)$ there is a change of variable which brings K into $U(T)$, i.e., there exists a homeomorphism $h : T \rightarrow T$ such that $f \circ h \in U(T)$ for all $f \in K$.*

The same authors, considered the classes

$$A_\varepsilon(T) = \left\{ x : \|x\|_{A_\varepsilon} = \sum_{k \in \mathbb{Z}} |\widehat{x}(k)| \varepsilon_{|k|} < \infty \right\},$$

where $\varepsilon = \{\varepsilon_n\}$ is a sequence of positive numbers that tends to zero, and proved (in 1981) the following theorem:

THEOREM E. *There exists a sequence $\varepsilon_n \rightarrow 0$ and a pair of functions in $C(T)$ such that there is no change of variables which will bring the pair into $A_\varepsilon(T)$.*

We notice that, if $\varepsilon_n \rightarrow 0$, then for every individually taken function there is a change of variable which brings it into A_ε (this follows from Theorem C).

In this paper we consider the classes $A_\varepsilon(T)$ and certain other classes of functions naturally characterised by the rate of decrease of Fourier coefficients and investigate if it is possible to bring compact families of functions in $C(T)$ into these classes.

We show (§ 2) that *under certain assumptions of regularity of the sequence ε the condition*

$$\sum_{n=1}^{\infty} \varepsilon_n/n < \infty$$

is necessary and sufficient in order that for every compact set $K \subset C(T)$ there is a change of variable which brings K into A_ε .

Here the sufficiency is a direct consequence of the following assertion (§ 1): *if $\lambda_n \rightarrow \infty$ and K is a compact subset of $C(T)$, then there exists a homeomorphism $h : T \rightarrow T$ such that*

$$|\widehat{f \circ h}(k)| = O(\lambda_{|k|}/|k|) \quad \forall f \in K.$$

It is not clear if one can put $\lambda_n \equiv 1$ in this assertion. However, it is impossible to attain the condition $|\widehat{f \circ h}(k)| = o(1/|k|)$, $\forall f \in K$. We show (§ 1) that *there exists a compact set such that there is no change of variable which will bring it into the class $\{x : |\widehat{x}(k)| = o(1/|k|)\}$* . This gives an answer to the problem posed in [1] (Russian p. 182, English p. 210) and in [2], § 4.2.

It follows from Theorem C that, for every function $f \in C(T)$, some superposition $x = f \circ h$ with a homeomorphism $h : T \rightarrow T$ satisfies

$$\sum |\widehat{x}(k)|^2 |k| < \infty. \quad (3)$$

We show (§ 3) that even for two functions it is, in general, impossible to attain (3) by a single homeomorphism: *if $f \in C(T)$ has unbounded variation*

on T , then there exists $g \in C(T)$ such that there is no change of variable which will bring the pair $\{f, g\}$ into the class defined by condition (3). Thus the answer to the problem posed in [3], p. 41, is negative. On the other hand, if $f \in C(T)$ is a function of bounded variation, then every pair $\{f, g\}$, $g \in C(T)$, can be brought to the indicated class.

We shall use notation $H^\omega(T)$ for the class of functions f on T that satisfy $\omega(\delta, f) = O(\omega(\delta))$, where

$$\omega(\delta, f) = \sup_{|t_1 - t_2| < \delta} |f(t_1) - f(t_2)|$$

is the uniform modulus of continuity of f , and $\omega(\delta)$ is a given increasing continuous function on $[0, \infty)$ with $\omega(0) = 0$.

§ 1. Estimates for $|\widehat{f \circ h}(k)|$, $f \in H^\omega$

THEOREM 1. Let $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Then, given any ω , there exists a homeomorphism h of T onto itself such that

$$|\widehat{f \circ h}(k)| = O(\lambda_{|k|}/|k|), \quad |k| \rightarrow \infty$$

for every $f \in H^\omega(T)$.

We define the integral modulus of continuity of a summable function x (of class $L(T)$) on T by

$$\omega_1(\delta, x) = \sup_{|\varepsilon| < \delta} \|x(\cdot + \varepsilon) - x(\cdot)\|_{L(T)}.$$

Theorem 1 is an immediate consequence of the well-known estimate $|\widehat{x}(k)| = O(\omega_1(1/|k|, x))$ and the following assertion:

LEMMA 1. Let $\lambda(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Then, given any ω , there exists a homeomorphism h of T onto itself such that

$$\omega_1(\delta, f \circ h) = O(\lambda(\delta)\delta), \quad \delta \rightarrow 0, \tag{4}$$

for every $f \in H^\omega(T)$.

PROOF. We use a modification of the method used to prove Theorem D (see [1], Theorem 4.2).

For an arbitrary set $E \subset T$ we denote its δ -neighbourhood by $(E)_\delta$. Let $L_\infty(T)$ denote the space of essentially bounded functions on T . Let $|E|$ denote the Lebesgue measure of a set E .

Let us prove first a simple lemma.

LEMMA 2. *Let $E \subset T$ be a closed set. Suppose that $x \in L_\infty(T)$ is a function which is constant on each interval complementary to E . Then*

$$\omega_1(\delta, x) \leq 2\|x\|_{L_\infty}|(E)_\delta|.$$

PROOF. For $\varepsilon > 0$ we have

$$\int_T |x(t + \varepsilon) - x(t)| dt = \int_{(E)_\varepsilon} |x(t + \varepsilon) - x(t)| dt \leq 2\|x\|_{L_\infty}|(E)_\varepsilon|.$$

Therefore $\omega_1(\delta, x) \leq 2\|x\|_{L_\infty}|(E)_\delta|$.

We now proceed to the proof of Lemma 1. Without loss of generality we may assume that $\lambda(\delta)$ is monotonic and that $\liminf \lambda(\delta)\delta = 0$. Consider a nowhere dense perfect set $E \subset [0, 2\pi]$ that contains the points 0 and 2π and satisfies $|(E)_\delta| \leq \lambda(\delta)\delta$, $\forall \delta > 0$. For an interval $I = (a, b) \subset [0, 2\pi]$ by $E(I)$ we denote the image of E under a homothetic mapping of $[0, 2\pi]$ onto $[a, b]$. It is easy to verify that

$$|(E(I))_\delta| \leq \lambda(\delta)\delta \quad \forall \delta > 0. \quad (5)$$

Let $N_k, k = 1, 2, \dots$, be a sequence of positive integers with $N_1 = 1$. Each such sequence defines sets $E_{kl}, k = 1, 2, \dots, l = 1, \dots, N_k$, as follows. We put $E_{11} = E = E((0, 2\pi))$. If the sets $E_{kl}, k = 1, 2, \dots, r, l = 1, \dots, N_k$ have already been defined, we put $E_{r+1l} = E(I_{r+1l}), l = 1, \dots, N_{r+1}$, where $I_{r+1l}, l = 1, 2, \dots$, are the intervals complementary to

$$\bigcup_{k=1}^r \bigcup_{l=1}^{N_k} E_{kl},$$

enumerated in the order of nonincreasing length.

It is clear that if the numbers N_k increase fast enough, then

$$\sup_l |I_{rl}| \rightarrow 0, \quad r \rightarrow \infty. \quad (6)$$

We choose a sequence $\{N_k\}$ and define sets E_{kl} , $k = 1, 2, \dots$, $l = 1, \dots, N_k$, so that (6) is satisfied.

Now we notice that if g is a continuous function on $[0, 2\pi]$, then we have an expansion

$$g \stackrel{L_\infty}{=} \sum_{k=1}^{\infty} \sum_{l=1}^{N_k} g_{kl}, \quad (7)$$

where each function g_{kl} is constant on the intervals complementary to the corresponding E_{kl} , the series converges in the L_∞ norm, and, in addition,

$$\|g_{11}\|_{L_\infty} \leq \|g\|_{L_\infty}, \quad \|g_{kl}\|_{L_\infty} \leq |g(I_{kl})|, \quad l = 1, \dots, N_k, \quad k > 1.$$

To prove this we proceed as follows. Consider a set of the form $E(I)$ and a function x continuous on I . Let $P_{E(I)}(x)$ stands for a function that takes constant value $(x(a) + x(b))/2$ on each interval $(a, b) \subset I$ complementary to $E(I)$ and vanishes at the other points of $[0, 2\pi]$. We put $g_{11} = P_{E_{11}}(g)$. If the functions g_{kl} , $l = 1, \dots, N_k$, $k = 1, \dots, r$ have already been constructed, we put

$$g_{r+1l} = P_{E_{r+1l}}\left(g - \sum_{k=1}^r \sum_{l=1}^{N_k} g_{kl}\right), \quad l = 1, \dots, N_{r+1}.$$

Continuing this process, we obtain (7). Indeed,

$$\left| \bigcup_{k=1}^{\infty} \bigcup_{l=1}^{N_k} E_{kl} \right| = 0$$

and, taking (6) into account, we have

$$\left\| g - \sum_{k=1}^r \sum_{l=1}^{N_k} g_{kl} \right\|_{L_\infty} \leq \sup_l \omega(|I_{r+1l}|, g) \rightarrow 0.$$

The rest of the properties of g_{kl} 's are obvious.

Now, for each set E_{kl} we fix a continuous increasing function h_{kl} that is constant on the intervals complementary to E_{kl} in $(0, 2\pi)$ and satisfies $h_{kl}(0) = 0$ and $h_{kl}(2\pi) = 2\pi$. Let

$$h = \sum_{k=1}^{\infty} \delta_k \sum_{l=1}^{N_k} h_{kl}.$$

We choose the numbers δ_k $k = 1, 2, \dots$, so that

$$\sum_{k=1}^{\infty} \delta_k N_k = 1, \quad (8)$$

and

$$\sum_{k=1}^{\infty} N_k \omega(\varepsilon_k) < \infty, \quad (9)$$

where

$$\varepsilon_k = 2\pi \sum_{s=k}^{\infty} \delta_s N_s.$$

It follows from (6) and (8) that h is a homeomorphism $T \rightarrow T$.

Note that

$$|h(I_{kl})| \leq \sum_{s=k}^{\infty} \delta_s \sum_{l=1}^{N_s} |h_{sl}(I_{kl})| \leq 2\pi \sum_{s=k}^{\infty} \delta_s N_s = \varepsilon_k, \quad l = 1, \dots, N_k, \quad k = 1, 2, \dots$$

Let $f \in H^\omega(T)$. Then for the function $g = f \circ h$ we have expansion (7), where

$$\|g_{11}\|_{L_\infty} \leq \|g\|_{L_\infty}, \quad \|g_{kl}\|_{L_\infty} \leq |f \circ h(I_{kl})| \leq \omega(\varepsilon_k, f), \quad l = 1, \dots, N_k, \quad k > 1.$$

Each function g_{kl} is constant on the intervals complementary to the corresponding set E_{kl} , so, by Lemma 2, taking (5) into account, we obtain

$$\omega_1(\delta, g_{11}) \leq 2\|g\|_{L_\infty} \lambda(\delta) \delta,$$

$$\omega_1(\delta, g_{kl}) \leq 2\omega(\varepsilon_k, f) \lambda(\delta) \delta \quad \forall \delta > 0, \quad k > 1, \quad l = 1, \dots, N_k,$$

which, together with (9), proves Lemma 1. Theorem 1 follows.

We note that for an individually taken function in $C(T)$ estimate (4) was established by B. S. Kashin (see [1], Russian p. 179, English p. 206).

Theorem 1 implies the following corollary:

COROLLARY 1. *For every class $H^\omega(T)$ there exists a change of variable which brings it into $\bigcap_{p>1} A_p(T)$, where $A_p(T)$ is the class of functions on T defined by*

$$x \in A_p(T) \Leftrightarrow \sum_{k \in Z} |\hat{x}(k)|^p < \infty.$$

This result was stated by A. M. Olevskiĭ in [1] (Russian p. 182, English, p. 210).

The following question related to Theorem 1 is open: Can we attain the estimate

$$|\widehat{f \circ h}(k)| = O(1/|k|) \quad \forall f \in H^\omega(T)?$$

It is also not clear whether it is possible to attain such an estimate for an arbitrary pair of functions in $C(T)$.

From Theorem C it is clear that if $f \in C(T)$, then for a certain homeomorphism $h : T \rightarrow T$ we have

$$|\widehat{f \circ h}(k)| = o(1/|k|). \quad (10)$$

Is it true that for every class $H^\omega(T)$ there is a homeomorphism h such that (10) holds for all $f \in H^\omega(T)$? This question was posed in [1] and [2]. Theorem 2 below gives a negative answer to this question. A similar question for pairs of functions is open.

THEOREM 2. *There exists a class $H^\omega(T)$ such that for every homeomorphism h of T onto itself, the condition*

$$|\widehat{f \circ h}(k)| = o(1/|k|) \quad \forall f \in H^\omega(T) \quad (11)$$

fails to be satisfied.

PROOF. Let $F \subset T$ be a perfect set. Suppose that E is a set of uniqueness. It is known (see [4], Chapter XIV, § 11) that if a function g is constant on the intervals complementary to F and satisfies

$$|\widehat{g}(k)| = o(1/|k|), \quad |k| \rightarrow \infty,$$

then g is equivalent to (i.e., coincides almost everywhere with) a constant function.

Let $\omega(\delta) = (\log(1/\delta))^{-\alpha}$, where $0 < \alpha < 1$. Let h be a homeomorphism $T \rightarrow T$. We shall construct a function $f \in H^\omega(T)$, $f \not\equiv \text{const}$, such that the superposition $f \circ h$ is constant on the intervals complementary to a certain

perfect set which is a set of uniqueness, thus the theorem will follow. Recall a result of Kahane and Salem (see [5], Chapter VII, § 8): if

$$\liminf_{\varepsilon \rightarrow 0} \frac{N_\varepsilon(F)}{\log(1/\varepsilon)} = 0, \quad (12)$$

where $N_\varepsilon(F)$ is the smallest number of intervals of length ε that cover F , then F is a set of uniqueness.

Let n_s be the positive integer closest to $2^{2^{s/\alpha}}$, $s = 1, 2, \dots$. Obviously we have

$$2^k / \log \prod_{s=1}^k n_s \rightarrow 0, \quad k \rightarrow \infty, \quad (13)$$

$$2^k \omega \left(2\pi \prod_{s=1}^k \frac{1}{2n_s + 1} \right) \geq \gamma > 0, \quad k = 1, 2, \dots, \quad (14)$$

where γ is independent of k .

Denote the homeomorphism inverse to h by h^{-1} . We partition the interval $[-\pi, \pi]$ into $2n_1 + 1$ pairwise nonoverlapping closed intervals of equal length. Enumerating them in order of succession on $[-\pi, \pi]$, we find, among the first n_1 intervals, the one, which we denote by I_0 , such that

$$|h^{-1}(I_0)| \leq 2\pi/n_1.$$

Similarly, among the last n_1 intervals we find an interval I_1 such that

$$|h^{-1}(I_1)| \leq 2\pi/n_1.$$

Suppose the closed intervals $I_{i_1 \dots i_s}$, $s = 1, \dots, k$, where $i_s = 0$ or 1 , have already been constructed. We partition $I_{i_1 \dots i_k}$ into $2n_{k+1} + 1$ closed intervals of equal length. Enumerating them in order of succession on $I_{i_1 \dots i_k}$, we find, among the first n_{k+1} and the last n_{k+1} intervals, the intervals which we denote by $I_{i_1 \dots i_k 0}$ and $I_{i_1 \dots i_k 1}$ respectively, such that

$$|h^{-1}(I_{i_1 \dots i_k i_{k+1}})| \leq \frac{1}{n_{k+1}} |h^{-1}(I_{i_1 \dots i_k})|, \quad i_{k+1} \in \{0; 1\}.$$

Thus we have a correspondence between tuples (i_1, \dots, i_k) , $k = 1, 2, \dots$, of zeros and ones, and closed intervals $I_{i_1 \dots i_k}$, $k = 1, 2, \dots$, with the following properties:

$$I_{i_1 \dots i_k i_{k+1}} \subset I_{i_1 \dots i_k}, \quad |h^{-1}(I_{i_1 \dots i_k})| \leq 2\pi / \prod_{s=1}^k n_s, \quad (15)$$

$$\begin{aligned} \inf\{|t_0 - t_1| : t_0 \in I_{i_1 \dots i_k 0}, t_1 \in I_{i_1 \dots i_k 1}\} &\geq \delta_{k+1} = \\ &= 2\pi \Big/ \prod_{s=1}^{k+1} (2n_s + 1), \quad k = 1, 2, \dots \end{aligned} \quad (16)$$

Let

$$E_k = \bigcup_{\alpha \in \{0;1\}^k} I_\alpha$$

(the union is taken over all k -tuples of zeros and ones). We put $E = \bigcap_{k=1}^{\infty} E_k$. Obviously, E is a nowhere dense perfect set. It follows from (15) that the set $F = h^{-1}(E)$ can be covered by 2^k intervals of length $2\pi / \prod_{s=1}^k n_s$, $k = 1, 2, \dots$. Taking (13) into account, we see that F satisfies (12), so $h^{-1}(E)$ is a set of uniqueness. To complete the proof of the theorem it remains to construct a function $f \in H^\omega(T)$, $f \not\equiv \text{const}$, which is constant on the intervals complementary to E .

For each closed interval $I = [a, b] \subset [-\pi, \pi]$, $a \neq b$, let ξ_I denote a function with the following properties: $\xi_I(t) = 0$ for $-\pi \leq t \leq a$, $\xi_I(t) = 1$ for $b \leq t \leq \pi$, ξ_I is linear and continuous on I . We put

$$g_n = 2^{-n} \sum_{\alpha \in \{0;1\}^n} \xi_{I_\alpha}.$$

It is easily seen that the sequence $\{g_n\}$ converges uniformly on $[-\pi, \pi]$. Let $g = \lim g_n$. We have $g(-\pi) = 0$, $g(\pi) = 1$. Let $f = \sin \pi g$. Then f is constant on each interval complementary to E , and $f \not\equiv \text{const}$.

Let us show that $f \in H^\omega(T)$. It suffices to verify that

$$|g(t_0) - g(t_1)| \leq \text{const } \omega(|t_0 - t_1|) \quad \forall t_0, t_1 \in E.$$

Let $t_0, t_1 \in E$. Then for a certain k we have $t_0, t_1 \in I_{i_1 \dots i_k}$ and at the same time $t_0 \in I_{i_1 \dots i_k 0}$, $t_1 \in I_{i_1 \dots i_k 1}$ (the case when $t_0 \in I_0$, $t_1 \in I_1$ can be omitted). Therefore,

$$|g_n(t_0) - g_n(t_1)| \leq 2^{-k} \quad \forall n \geq k,$$

whence

$$|g(t_0) - g(t_1)| \leq 2^{-k}.$$

By (16) we have $|t_0 - t_1| \geq \delta_{k+1}$, so, taking (14) into account, we obtain

$$|g(t_0) - g(t_1)| \leq 2 \cdot 2^{-(k+1)} \leq (2/\gamma)\omega(\delta_{k+1}) \leq (2/\gamma)\omega(|t_0 - t_1|).$$

The theorem is proved.

We note that the function f constructed in the proof is of bounded variation on T .

It is not clear for what precisely ω 's the conclusion of Theorem 2 holds. In the proof we used $\omega(\delta) = (\log(1/\delta))^{-\alpha}$, $0 < \alpha < 1$. Similarly one can show that if

$$\lim \omega(\delta) \log(1/\delta) = \infty,$$

then it is impossible to attain (11). Possibly the same holds for all ω unless $\omega(\delta) = O(\delta)$.

§ 2. The classes A_ε

As we mentioned in Introduction, for every function in $C(T)$ there is a change of variable which brings it into any given class $A_\varepsilon(T)$. This result does not extend to compact families of functions. Moreover there exists a sequence $\varepsilon_n \rightarrow 0$ such that, in general, there is no single change of variable which will bring two functions into A_ε . We do not know what conditions imposed on ε are necessary and sufficient in order that for every pair of (real-valued) continuous functions there is a change of variable that brings the pair into A_ε .

From Theorem 1, we immediately obtain the following corollary.

COROLLARY 2. *Let*

$$\sum_{n=1}^{\infty} \varepsilon_n/n < \infty. \tag{17}$$

Then for every class $H^\omega(T)$ there is a change of variable which brings it into $A_\varepsilon(T)$, i.e., there exists a homeomorphism h of the circle T such that $f \circ h \in A_\varepsilon(T)$ for all $f \in H^\omega(T)$

For the proof it suffices to choose a sequence $\{\lambda_n\}$, $\lambda_n \rightarrow \infty$, with

$$\sum_{n=1}^{\infty} \lambda_n \varepsilon_n/n < \infty,$$

and apply Theorem 1.

We say that a sequence ε is regular if it is non-increasing, and $\{n\varepsilon_n\}$ is non-decreasing.

The following theorem shows that, for regular sequences, condition (17) is also necessary in order that for every class $H^\omega(T)$ there is a change of variable which brings $H^\omega(T)$ into A_ε .

THEOREM 3. *Suppose that the sequence ε is regular and*

$$\sum_{n=1}^{\infty} \varepsilon_n/n = \infty. \quad (18)$$

Then there exists a class $H^\omega(T)$ such that there is no change of variable which will bring it into $A_\varepsilon(T)$, i.e., for every homeomorphism $h : T \rightarrow T$ there is an $f \in H^\omega(T)$ such that $f \circ h \notin A_\varepsilon(T)$.

PROOF. We say that a function f is of class $H_{\text{loc}(0)}^\omega$ if there is an interval $I \subseteq T$, containing $\{0\}$, such that

$$\sup_{\substack{|t_1-t_2|<\delta \\ t_1, t_2 \in I}} |f(t_1) - f(t_2)| = O(\omega(\delta)), \quad \delta \rightarrow 0.$$

It is known that if a function $f \in A(T)$ is monotonic in a neighborhood of a point, then in a certain neighborhood of this point the modulus of continuity of f is logarithmic at worst (Katznelson; see [5], Chapter II, § 12). Similar result holds for classes A_ε provided that ε_n tends to zero sufficiently slowly (see [1], Lemma 4.3). The following lemma shows that this result is valid under the assumption that (18) is satisfied and ε is regular.

LEMMA 3. *Under the assumptions of the theorem on the sequence ε there exists a function ω_ε satisfying $\omega_\varepsilon(\delta) \downarrow 0$ as $\delta \downarrow 0$, such that if $f \in A_\varepsilon(T) \cap C(T)$ and f is monotonic in a neighborhood of zero, then $f \in H_{\text{loc}(0)}^{\omega_\varepsilon}$.*

PROOF. We identify T with the interval $[-\pi, \pi]$. Let $\gamma(\theta)$ be the function on $(0, \pi]$ that takes value $n\varepsilon_n$ for $\theta \in (\pi/(n+1), \pi/n]$, $n = 1, 2, \dots$. The function $\gamma(\theta)$ defined in this way increases as θ decreases to zero, so

$$\begin{aligned} \sup_{0 \leq \alpha \leq \pi} \left| \int_0^\alpha \gamma(\theta) \sin k\theta d\theta \right| &\leq \left| \int_0^{\pi/|k|} \gamma(\theta) \sin k\theta d\theta \right| \leq \\ &\leq |k| \left| \int_0^{\pi/|k|} \gamma(\theta) \theta d\theta \right| = |k| \sum_{n \geq |k|} n\varepsilon_n \int_{\pi/(n+1)}^{\pi/n} \theta d\theta \leq \pi^2 \varepsilon_{|k|}, \quad k \neq 0. \end{aligned} \quad (19)$$

We now follow the method used to prove Lemma 4.3 in [1]. Note that if $f \in A_\varepsilon(T) \cap C(T)$ and $0 < \delta < \delta_0 < \pi$, then

$$\left| \int_\delta^{\delta_0} (f(t+\theta) - f(t-\theta))\gamma(\theta)d\theta \right| \leq 4\pi^2 \|f\|_{A_\varepsilon}, \quad t \in T. \quad (20)$$

Indeed, if f is a trigonometric polynomial, then, using (19), we obtain

$$\begin{aligned} & \left| \int_\delta^{\delta_0} (f(t+\theta) - f(t-\theta))\gamma(\theta)d\theta \right| = \\ & = 2 \left| \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{ikt} \int_\delta^{\delta_0} \gamma(\theta) \sin k\theta d\theta \right| \leq 4\pi^2 \|f\|_{A_\varepsilon}. \end{aligned}$$

In the general case one should approximate f by its Fejér sums.

Let f be monotonic in a $2\delta_0$ -neighborhood of zero. Then, for every t in the δ_0 -neighborhood of zero and for every δ with $0 < \delta < \delta_0$ we obtain from (20)

$$|f(t+\delta) - f(t-\delta)| \int_\delta^{\delta_0} \gamma(\theta)d\theta \leq \left| \int_\delta^{\delta_0} (f(t+\theta) - f(t-\theta))\gamma(\theta)d\theta \right| \leq 4\pi^2 \|f\|_{A_\varepsilon}.$$

Therefore

$$|f(t+\delta) - f(t)| \int_\delta^{\delta_0} \gamma(\theta)d\theta = O(1)$$

uniformly with respect to t , $|t| < \delta_0$. It remains only to notice that

$$\int_\delta^\pi \gamma(\theta)d\theta \uparrow \infty$$

as $\delta \downarrow 0$, and to put

$$\omega_\varepsilon(\delta) = 1 \Big/ \int_\delta^\pi \gamma(\theta)d\theta.$$

The lemma is proved.

Let us now show that the smoothness of all functions in $H^\omega(T)$ which are monotonic in a neighborhood of zero cannot be improved even locally.

LEMMA 4. *Let $\omega(2\delta) \leq 2\omega(\delta)$ for all $\delta > 0$. Let $\bar{\omega}(\delta) = o(\omega(\delta))$ as $\delta \rightarrow 0$, and let h be a homeomorphism $T \rightarrow T$ with $h(0) = 0$. Then there exists a function $f \in H^\omega(T)$, monotonic in a neighborhood of zero, such that $f \circ h \notin H_{\text{loc}(0)}^{\bar{\omega}}$.*

PROOF. Similarly to what we did to prove Theorem 2, for each closed interval $[a, b] \subset [-\pi, \pi]$ consider a function ξ_I continuous on $[-\pi, \pi]$, such that $\xi_I(t) = 0$ for $-\pi \leq t \leq a$, $\xi_I(t) = 1$ for $b \leq t \leq \pi$, and ξ_I is linear on I .

We fix a positive sequence λ_n , $n = 1, 2, \dots$, $\lambda_n \rightarrow 0$, such that

$$\frac{\bar{\omega}(1/n)}{\omega(\lambda_n/n)} \rightarrow 0, \quad n \rightarrow \infty.$$

Choose a sequence of positive integers $\{n_k\}$ with the following properties

$$\sum_{k=1}^{\infty} \lambda_{n_k} < \pi/20,$$

$$\omega(\lambda_{n_{k+1}}/n_{k+1}) \leq \frac{1}{2}\omega(\lambda_{n_k}/n_k), \quad k = 1, 2, \dots$$

Let $d_k = \lambda_{n_k}/n_k$. Then

$$\bar{\omega}(1/n_k) = o(\omega(d_k)), \quad k \rightarrow \infty, \quad (21)$$

$$\sum_{k=1}^{\infty} 20n_k d_k = d < \pi, \quad (22)$$

$$\omega(d_{k+1}) \leq \frac{1}{2}\omega(d_k), \quad k = 1, 2, \dots \quad (23)$$

Choose points a_k , $k = 1, 2, \dots$, in the interval $[0, d]$ so that $0 < a_{k+1} < a_k$, $a_k \rightarrow 0$, and $|a_k - a_{k+1}| = 20n_k d_k$; this is possible since (22). We partition each interval $[a_{k+1}, a_k]$ into $20n_k$ closed intervals of length d_k .

Let h be a homeomorphism $T \rightarrow T$ with $h(0) = 0$. For $k = 1, 2, \dots$ we exclude from the intervals that form the partition of $[a_{k+1}, a_k]$ the left-hand one and find among the remaining ones an interval I_k such that

$$|h^{-1}(I_k)| \leq 2\pi/(20n_k - 1). \quad (24)$$

By our construction, if $k_1 < k_2$, then there is an interval of length d_{k_1} between I_{k_1} and I_{k_2} .

We put

$$g = \sum_{k=1}^{\infty} \omega(d_k) \xi_{I_k}$$

and define a function $f \in C(T)$ as follows: $f(t) = g(t)$ for $-\pi \leq t \leq d$, $f(\pi) = 0$, and f is linear on $[d, \pi]$. Obviously f is monotonic in a neighborhood of zero.

Let us show that $f \in H^\omega(T)$. It suffices to verify that

$$|g(t_1) - g(t_2)| \leq \text{const} \cdot \omega(|t_1 - t_2|)$$

for $t_1, t_2 \in \bigcup_k I_k$. Assume first that $t_1, t_2 \in I_k$, $t_1 \neq t_2$. Then, choosing a positive integer p such that

$$2^p \leq \frac{d_k}{|t_1 - t_2|} < 2^{p+1},$$

we have

$$\begin{aligned} |g(t_1) - g(t_2)| &= \frac{\omega(d_k)}{d_k} |t_1 - t_2| \leq \omega(2^{p+1}|t_1 - t_2|) 2^{-p} \leq \\ &\leq 2^{p+1} \omega(|t_1 - t_2|) 2^{-p} = 2\omega(|t_1 - t_2|). \end{aligned}$$

Assume now that $t_1 \in I_{k_1}$ and $t_2 \in I_{k_2}$, $k_1 < k_2$. Then $|t_1 - t_2| \geq d_{k_1}$, and it follows from (23) that

$$|g(t_1) - g(t_2)| \leq \sum_{k_1 \leq k \leq k_2} \omega(d_k) \leq 2\omega(d_{k_1}) \leq 2\omega(|t_1 - t_2|).$$

Let us show that $f \circ h \notin H_{\text{loc}(0)}^\omega$. Assuming the contrary, from (24) we obtain

$$\omega(d_k) = |f(I_k)| = |f \circ h(h^{-1}(I_k))| = O(\overline{\omega}(|h^{-1}(I_k)|)) = O(\overline{\omega}(1/n_k)), \quad k \rightarrow \infty,$$

which contradicts (21). The lemma is proved.

We shall now complete the proof of the theorem. Let ω be an increasing continuous function on $[0, \infty)$ with $\omega(0) = 0$, $\omega(2\delta) \leq 2\omega(\delta) \forall \delta > 0$, and $\omega_\varepsilon(\delta) = o(\omega(\delta))$, where ω_ε is the function from Lemma 3. For example these conditions hold for

$$\omega(\delta) = \sup_{|t_1 - t_2| \leq \delta} |\sqrt{\omega_\varepsilon(t_1)} - \sqrt{\omega_\varepsilon(t_2)}|.$$

Suppose that for some homeomorphism h of T onto itself we have $f \circ h \in A_\varepsilon(T)$ for all $f \in H^\omega(T)$. We may assume that $h(0) = 0$. By Lemma 4, there exists a function $f \in H^\omega(T)$, monotonic in a neighborhood of zero, such that $f \circ h \notin H_{\text{loc}(0)}^{\omega_\varepsilon}$. But, since $f \circ h$ is monotonic in a neighborhood of zero, it follows from Lemma 3 that $f \circ h \in H_{\text{loc}(0)}^{\omega_\varepsilon}$. The contradiction proves the theorem.

§ 3. Sobolev classes

Let $W_2^\lambda(T)$ be the class of all functions x with

$$\|x\|_{W_2^\lambda} = \left(\sum_{k \in Z} (|\hat{x}(k)| |k|^\lambda)^2 \right)^{1/2} < \infty.$$

Theorem 1 implies the following corollary:

COROLLARY 3. *For every class $H^\omega(T)$ there exists a change of variable which brings it into $\bigcap_{\lambda < 1/2} W_2^\lambda(T)$, i.e., there exists a homeomorphism h of the circle T onto itself such that $f \circ h \in \bigcap_{\lambda < 1/2} W_2^\lambda(T)$ for all $f \in H^\omega(T)$.*

Let us recall that for every function in $C(T)$ there is a change of variable which brings it into $W_2^{1/2}$. Is it true that for every class H^ω there exists a change of variable which brings it into $W_2^{1/2}$? The answer to this question posed in [3, p.41] is negative. Moreover the following theorem holds.

THEOREM 4. *Let $f \in C(T)$. The following conditions are equivalent:*

- (i) *For every function $g \in C(T)$ there exists a change of variable which brings the pair $\{f, g\}$ into $W_2^{1/2}(T)$, i.e., there is a homeomorphism $h : T \rightarrow T$ such that $f \circ h \in W_2^{1/2}$ and $g \circ h \in W_2^{1/2}$.*
- (ii) *f is of bounded variation on T .*

PROOF. Note that the obvious estimate

$$c_1 |k| \leq \int_0^1 \left(\frac{\sin k\delta}{\delta} \right)^2 d\delta \leq c_2 |k|, \quad k \in Z$$

(where $c_1, c_2 > 0$ are independent of k) and the identity

$$\frac{1}{2\pi} \int_T |x(t + \delta) - x(t - \delta)|^2 dt = 4 \sum_{k \in Z} |\hat{x}(k)|^2 \sin^2 k\delta$$

imply the equivalence of the seminorms $\|\cdot\|_{W_2^{1/2}}$ and $\|\cdot\|$, where

$$\|x\| = \left(\int_0^1 \frac{1}{\delta^2} \int_T |x(t+\delta) - x(t-\delta)|^2 dt d\delta \right)^{1/2}.$$

We shall use this fact later.

By $\text{Var}(x, E)$ we denote the variation of a function $x(t)$ on a set $E \subseteq T$.

1) (i) \Rightarrow (ii). Let $f \in C(T)$. Under the assumption that $\text{Var}(f, T) = \infty$ we shall construct a function $g \in C(T)$ such that there is no single change of variable which will bring both f and g into $W_2^{1/2}$. Thus the implication (i) \Rightarrow (ii) will be proved.

LEMMA 5. *There exists a monotonic sequence $t_k \in T$, $k = 1, 2, \dots$, such that*

$$\sum_{k=1}^{\infty} |f(t_{k+1}) - f(t_k)| = \infty.$$

PROOF. Note that there exists a point $\theta \in T$ such that f has infinite variation in every neighborhood of θ . Indeed, otherwise each point $t \in T$ would have a neighbourhood U_t such that $\text{Var}(f, U_t) < \infty$, and choosing a finite covering of the circle from the family $\{U_t, t \in T\}$ we would obtain $\text{Var}(f, T) < \infty$ which contradicts the assumption.

Fix $\theta \in T$ with the indicated property. Then either $\text{Var}(f, (\theta', \theta)) = \infty$ for every open interval (θ', θ) , $\theta' < \theta$, or $\text{Var}(f, (\theta, \theta')) = \infty$ for every open interval (θ, θ') , $\theta' > \theta$.

Consider the first case (the second one is similar). Since $\text{Var}(f, (-\pi, \theta)) = \infty$, one can find points t_k , $k = 1, \dots, n_1$, $-\pi < t_1 < \dots < t_k < t_{k+1} < \dots < t_{n_1} < \theta$, such that

$$\sum_{k=1}^{n_1-1} |f(t_{k+1}) - f(t_k)| > 1.$$

Assume that the points $t_1 < \dots < t_{n_s} < \theta$ have already been defined. Since $\text{Var}(f, (t_{n_s}, \theta)) = \infty$, one can find points t_k , $k = n_s + 1, \dots, n_{s+1}$, $t_{n_s} < t_{n_s+1} < \dots < t_{n_{s+1}} < \theta$ such that

$$\sum_{k=n_s+1}^{n_{s+1}-1} |f(t_{k+1}) - f(t_k)| > 1.$$

Continuing this process, we obtain the required sequence.

LEMMA 6. *There exist a function $g \in C(T)$ and a sequence of functions g_n , $n = 1, 2, \dots$, with the following properties:*

$$|g_n(t_1) - g_n(t_2)| \leq |g(t_1) - g(t_2)| \quad \forall t_1, t_2, \quad (25)$$

$$\text{Var}(g_n, T) < \infty, \quad \forall n, \quad (26)$$

$$\sup_n \left| \int_T f(t) dg_n(t) \right| = \infty. \quad (27)$$

PROOF. Let $\{t_k\}$ be the sequence from Lemma 5. Put $M^+ = \{k : f(t_{k+1}) - f(t_k) > 0\}$, $M^- = \{k : f(t_{k+1}) - f(t_k) < 0\}$. Then at least one of the sums

$$\sum_{k \in M^+} |f(t_{k+1}) - f(t_k)|, \quad \sum_{k \in M^-} |f(t_{k+1}) - f(t_k)|$$

is infinite. We proceed with our construction under the assumptions that $t_1 < t_2 < \dots < t_k < t_{k+1} < \dots$ and

$$\sum_{k \in M^+} |f(t_{k+1}) - f(t_k)| = \infty$$

(the other cases are similar).

Let k_j , $j = 1, 2, \dots$, be the sequence of numbers that form M^+ . Put $a_j = t_{k_j}$, $b_j = t_{k_j+1}$, $j = 1, 2, \dots$. The intervals (a_j, b_j) are pairwise disjoint and

$$f(b_j) - f(a_j) > 0, \quad \sum_j (f(b_j) - f(a_j)) = \infty. \quad (28)$$

Choose a sequence of positive numbers γ_j , $j = 1, 2, \dots$, that decreases to zero and satisfies

$$\sum_j \gamma_j (f(b_j) - f(a_j)) = \infty. \quad (29)$$

Choose also a sequence $\{\varepsilon_j\}$ such that $0 < 2\varepsilon_j < b_j - a_j$ and

$$\omega(\varepsilon_j, f) < 2^{-j}. \quad (30)$$

For $j = 1, 2, \dots$ we define functions λ_j as follows: $\lambda_j \in C(T)$, $\lambda_j(t) = 0$ for $t \notin (a_j, b_j)$, $\lambda_j(t) = \gamma_j$ for $t \in (a_j + \varepsilon_j, b_j - \varepsilon_j)$, and λ_j is linear on $(a_j, a_j + \varepsilon_j)$ and $(b_j - \varepsilon_j, b_j)$.

Let

$$g(t) = \sum_{j=1}^{\infty} \lambda_j(t)$$

Obviously $g \in C(T)$. Let

$$g_n(t) = \max\{\gamma_n, g(t)\}.$$

It is easily verified that (25) and (26) hold.

Let us show that (27) holds. Note that the set $\{t : \gamma_n < \lambda_j(t) < \gamma_j\}$ is empty if $j \geq n$, whereas if $1 \leq j < n$ it consists of two intervals I_{jn}^+ and I_{jn}^- of equal length on which g increases and decreases respectively. Note also that

$$|I_{jn}^{\pm}| \leq \varepsilon_j, \quad (31)$$

$$\lim_n |I_{jn}^{\pm}| = \varepsilon_j \quad \forall j. \quad (32)$$

We have

$$\begin{aligned} \int_T f(t) dg_n(t) &= \sum_{j=1}^{n-1} \left(\int_{I_{jn}^+} f(t) \frac{\gamma_j}{\varepsilon_j} dt - \int_{I_{jn}^-} f(t) \frac{\gamma_j}{\varepsilon_j} dt \right) = \\ &= \sum_{j=1}^{n-1} \frac{\gamma_j}{\varepsilon_j} \left[\int_{I_{jn}^+} (f(t) - f(a_j)) dt + f(a_j) |I_{jn}^+| - \int_{I_{jn}^-} (f(t) - f(b_j)) dt - f(b_j) |I_{jn}^-| \right]. \end{aligned}$$

Taking (31) and (30) into account, we obtain

$$\begin{aligned} &\left| \int_T f(t) dg_n(t) + \sum_{j=1}^{n-1} \gamma_j (f(b_j) - f(a_j)) \frac{|I_{jn}^{\pm}|}{\varepsilon_j} \right| \leq \\ &\leq \sum_{j=1}^{n-1} \frac{\gamma_j}{\varepsilon_j} \left(\int_{I_{jn}^+} |f(t) - f(a_j)| dt + \int_{I_{jn}^-} |f(t) - f(b_j)| dt \right) \leq \\ &\leq \sum_{j=1}^{n-1} 2\gamma_j \omega(\varepsilon_j, f) = O(1), \quad n \rightarrow \infty. \end{aligned} \quad (33)$$

Suppose that (27) does not hold. Then (see (33))

$$\sum_{j=1}^{n-1} \gamma_j(f(b_j) - f(a_j)) \frac{|I_{jn}^\pm|}{\varepsilon_j} = O(1). \quad (34)$$

Since the terms in (34) are positive (see (28)), we have for $m < n$

$$\sum_{j=1}^m \gamma_j(f(b_j) - f(a_j)) \frac{|I_{jn}^\pm|}{\varepsilon_j} \leq \text{const}. \quad (35)$$

Using (32) and taking the limit in (35), we obtain

$$\sum_{j=1}^m \gamma_j(f(b_j) - f(a_j)) \leq \text{const}.$$

Since m is arbitrary, this contradicts (29). The lemma is proved.

Let g be a function as in Lemma 6. Let us show that there is no change of variable which will bring the pair $\{f, g\}$ into $W_2^{1/2}$. Suppose that, on the contrary, $f \circ h \in W_2^{1/2}$ and $g \circ h \in W_2^{1/2}$ for a certain homeomorphism $h : T \rightarrow T$. Then using (25) and the equivalence of the seminorms $\|\cdot\|_{W_2^{1/2}}$ and $\|\cdot\|$ we see that $g_n \circ h \in W_2^{1/2}$ for all n , and

$$\|g_n \circ h\|_{W_2^{1/2}} \leq \text{const} \cdot \|g \circ h\|_{W_2^{1/2}}. \quad (36)$$

Note now, that if $x, y \in W_2^{1/2}(T) \cap C(T)$ and y is a function of bounded variation, then

$$\frac{1}{2\pi} \left| \int_T x(t) dy(t) \right| \leq \|x\|_{W_2^{1/2}} \|y\|_{W_2^{1/2}}.$$

(This is obvious if x is a trigonometric polynomial; in the general case one should approximate x by Fejér sums.) Thus, from (36), taking (26) into account, we have

$$\frac{1}{2\pi} \left| \int_T f(t) dg_n(t) \right| = \frac{1}{2\pi} \left| \int_T f \circ h(t) dg_n \circ h(t) \right| \leq \|f \circ h\|_{W_2^{1/2}} \|g_n \circ h\|_{W_2^{1/2}} = O(1).$$

This contradicts (27). The implication (i) \Rightarrow (ii) is proved.

2) (ii) \Rightarrow (i). The method of the proof of Theorem A that uses a conformal mapping of the disk onto a suitable domain (see [1], Proof of Theorem 3.1) admits the following modification that allows to bring an arbitrarily given function $g \in C(T)$ into $W_2^{1/2}$. Assuming that $g(t) \geq \gamma > 0$ for all t (this does not restrict generality) we put $Q(t) = g(t)e^{it}$. Obviously, when t runs over $[-\pi, \pi]$ the point $Q(t)$ describes a simple closed curve Γ in the complex plane. Let D be a domain bounded by this curve, and let Φ be a conformal mapping of the disk $|z| < 1$ onto D . Then Φ extends continuously to the circle $|z| = 1$, and provides a homeomorphism of the circle onto Γ . Thus the function $\varphi(t) = \Phi(e^{it})$ has the form $\varphi = Q \circ h$ where h is a homeomorphism $T \rightarrow T$.

It is well known that $\pi \sum_{n \geq 0} |\widehat{\varphi}(n)|^2 n$ is the area of D . Therefore $Q \circ h \in W_2^{1/2}(T)$. Using the equivalence of the seminorms $\|\cdot\|_{W_2^{1/2}}$ and $\|\cdot\|$, we obtain $|Q \circ h| \in W_2^{1/2}$ and it remains to note that $|Q \circ h| = g \circ h$. This modification of the Pál–Bohr theorem was found by A. M. Olevskii (personal communication).

We now note that if $x \in W_2^{1/2} \cap C(T)$ and $x(t) \geq \gamma > 0$ for all t , then $1/x \in W_2^{1/2}$; this follows from the equivalence of the seminorms. This equivalence implies also that if two continuous complex-valued functions are of class $W_2^{1/2}$ then their product is in $W_2^{1/2}$. So, in the construction described above, we obtain in addition that

$$e^{ih} = \frac{1}{|Q \circ h|} Q \circ h \in W_2^{1/2}.$$

We have thus proved the following lemma.

LEMMA 7. *For every pair $\{g(t), e^{it}\}$, where $g \in C(T)$, there is a change of variable which brings it into $W_2^{1/2}(T)$, i.e., there exists a homeomorphism $h : T \rightarrow T$ such that $g \circ h \in W_2^{1/2}(T)$ and $e^{ih} \in W_2^{1/2}(T)$.*

We now complete the proof of the implication (ii) \Rightarrow (i). Suppose that (ii) holds and g is an arbitrary function in $C(T)$. It is easily seen that a function, which is continuous and of bounded variation on the circle, can be turned into a Lipschitz function by an appropriate change of variable. Fix a homeomorphism $\psi : T \rightarrow T$ such that

$$|f \circ \psi(t_1) - f \circ \psi(t_2)| \leq \text{const} \cdot |e^{it_1} - e^{it_2}| \quad \forall t_1, t_2.$$

Applying Lemma 7 to the function $g \circ \psi$, we obtain a homeomorphism $h : T \rightarrow T$ such that $g \circ \psi \circ h \in W_2^{1/2}(T)$ and

$$e^{ih} \in W_2^{1/2}(T). \quad (37)$$

It remains only to observe that, taking account of the equivalence of the seminorms $\|\cdot\|_{W_2^{1/2}}$ and $\|\cdot\|$, from (37) and the inequality

$$|f \circ \psi \circ h(t_1) - f \circ \psi \circ h(t_2)| \leq \text{const} \cdot |e^{ih(t_1)} - e^{ih(t_2)}| \quad \forall t_1, t_2$$

it follows that $f \circ \psi \circ h \in W_2^{1/2}(T)$.

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